1 Continuum Mechanics: Review of Principles

1.1 Strain and Rate-of-Strain Tensor

1.1.1 Strain Tensor

1.1.1.1 Phenomenological Definitions

Phenomenological definitions of strain are first presented in the following examples.

1.1.1.1.1 Extension (or Compression)

In extension, a volume element of length l is elongated by Δl in the x direction, as illustrated by Figure 1.1. The strain can be defined, from a phenomenological point of view, as $\varepsilon = \Delta l / l$.

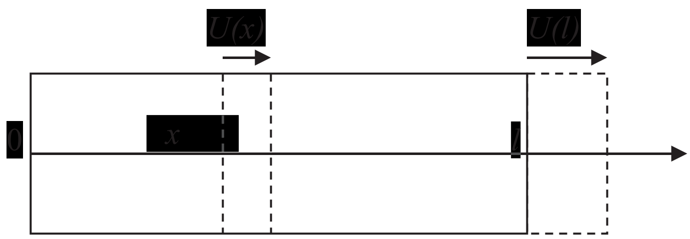

Figure 1.1 Strain in extension

For a homogeneous deformation of the volume element, the displacement U on the x-axis is $U(x) = \Delta l \dfrac{x}{l}$, and $\dfrac{dU}{dx} = \dfrac{\Delta l}{l}$. Hence another definition of the strain is $\varepsilon = \dfrac{dU}{dx}$.

1.1.1.1.2 Pure Shear

A volume element of square section $h \times h$ in the x-y plane is sheared by a value a in the x-direction, as shown in Figure 1.2. Intuitively, the strain may be defined as $\gamma = a/h$. For a homogeneous deformation of the volume element, the displacement (U, V) of point $M(x, y)$ is

$$U(y) = a\frac{y}{h};V = 0 \tag{1.1}$$

Hence, another possible definition of the strain is $\gamma = \dfrac{dU}{dy}$.

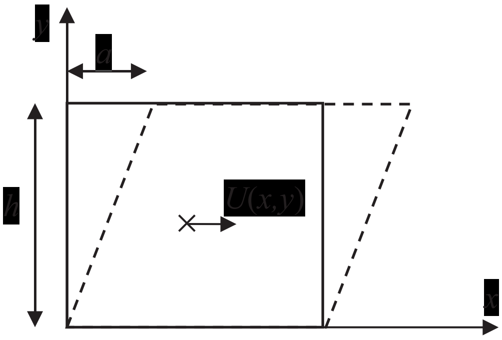

Figure 1.2 Strain in pure shear

1.1.1.2 Displacement Gradient

More generally, any strain in a continuous medium is defined through a field of the displacement vector $\mathbf{U}(x, y, z)$ with coordinates

$$U(x, y, z), \quad V(x, y, z), \quad W(x, y, z)$$

The intuitive definitions of strain make use of the derivatives of U, V, and W with respect to x, y, and z, that is, of their gradients. For a three-dimensional flow, the material can be deformed in nine different ways: three in extension (or compression) and six in shear. Therefore, it is natural to introduce the nine components of the displacement gradient tensor $\nabla\mathbf{U}$:

$$\nabla\mathbf{U} = \begin{bmatrix} \dfrac{\partial U}{\partial x} & \dfrac{\partial U}{\partial y} & \dfrac{\partial U}{\partial z} \\ \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial y} & \dfrac{\partial V}{\partial z} \\ \dfrac{\partial W}{\partial x} & \dfrac{\partial W}{\partial y} & \dfrac{\partial W}{\partial z} \end{bmatrix} \tag{1.2}$$

This notion of displacement gradient applied to the two previous deformations presented in Section 1.1.1.1 leads to the following expressions:

- Extension deformation:

$$\nabla \mathbf{U} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.3}$$

- Shear deformation:

$$\nabla \mathbf{U} = \begin{bmatrix} 0 & \gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.4}$$

If this notion is applied to a volume element that has rotated θ degrees without being deformed, as shown in Figure 1.3, the displacement vector can be written as

$$\mathbf{U} = \left| \begin{array}{l} U(x,y) = x(\cos\theta - 1) - y\sin\theta \\ V(x,y) = x\sin\theta + y(\cos\theta - 1) \end{array} \right. \tag{1.5}$$

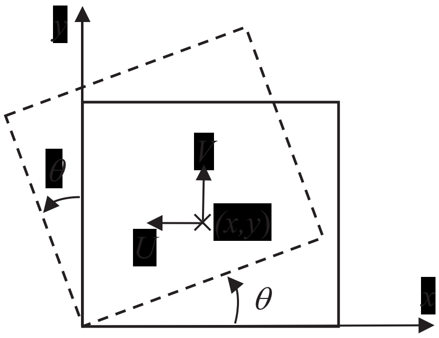

Figure 1.3 Rigid rotation

For a very small value of θ: $U(x,y) \approx -y\theta$ (1.6)

$V(x,y) \approx x\theta$

hence

$$\nabla \mathbf{U} = \begin{bmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.7}$$

It is obvious from this result that $\nabla \mathbf{U}$ cannot physically describe the strain of the material since it is not equal to zero when the material is under rigid rotation without being deformed.

1.1.1.3 Deformation or Strain Tensor ε

To obtain a tensor that physically represents the local deformation, we must make the tensor $\nabla\mathbf{U}$ symmetrical, as follows:

- Write the transposed tensor (symmetry with respect to the principal diagonal); the transposed deformation tensor is

$$(\nabla\mathbf{U})^t = \begin{bmatrix} \frac{\partial U}{\partial x} & \frac{\partial V}{\partial x} & \frac{\partial W}{\partial x} \\ \frac{\partial U}{\partial y} & \frac{\partial V}{\partial y} & \frac{\partial W}{\partial y} \\ \frac{\partial U}{\partial z} & \frac{\partial V}{\partial z} & \frac{\partial W}{\partial z} \end{bmatrix} \tag{1.8}$$

- Write the half sum of the two tensors, each transposed with respect to the other:

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\mathbf{U} + (\nabla\mathbf{U})^t\right) \tag{1.9}$$

$$\text{or} \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \tag{1.10}$$

where U_i stands for U, V, or W and x_i for x, y, or z.

Let us now reexamine the three previous cases:

- In *extension* (or compression):

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.11}$$

The deformation tensor **ε** is equal to the displacement gradient tensor $\nabla\mathbf{U}$.

- In *pure shear*:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} 0 & \frac{1}{2}\gamma & 0 \\ \frac{1}{2}\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.12}$$

The tensor **ε** is symmetric, whereas $\nabla\mathbf{U}$ is not. We see that pure shear is physically imposed in a nonsymmetrical manner with respect to x and y; however, the strain experienced by the material is symmetrical.

- In *rigid rotation*:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.13}$$

The definition of $\boldsymbol{\varepsilon}$ is such that the deformation is nil in rigid rotation; it is physically satisfactory, whereas the use of $\nabla \mathbf{U}$ for the deformation is not correct.

As a general result, the tensor $\boldsymbol{\varepsilon}$ is always symmetrical; that is, it contains only six independent components:

- three in extension or compression: ε_{xx}, ε_{yy}, ε_{zz}
- three in shear: $\varepsilon_{xy} = \varepsilon_{yx}$, $\varepsilon_{yz} = \varepsilon_{zy}$, $\varepsilon_{zx} = \varepsilon_{xz}$

Important Remarks

(a) The definition of the tensor $\boldsymbol{\varepsilon}$ used here is a simplified one. One can show rigorously that the strain tensor in a material is mathematically described by the tensor $\boldsymbol{\Delta}$ (Salençon, 1988):

$$\Delta_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} + \sum_k \frac{\partial U_k}{\partial x_i}\frac{\partial U_k}{\partial x_j} \right) = \varepsilon_{ij} + \frac{1}{2}\sum_k \frac{\partial U_k}{\partial x_i}\frac{\partial U_k}{\partial x_j} \tag{1.14}$$

This definition of the tensor $\boldsymbol{\varepsilon}$ is valid only if the terms $\partial U_i / \partial x_j$ are small. So the expressions for the tensor written above are usable only if $\varepsilon, \gamma, \theta$, and so on are small (typically less than 5%). This condition is not generally satisfied for the flow of polymer melts. As will be shown, in those cases, we will use the rate-of-strain tensor $\dot{\boldsymbol{\varepsilon}}$.

(b) The deformation can also be described by following the homogeneous deformation of a continuum media with time. The Cauchy tensor is then used, defined by

$$\mathbf{C} = \mathbf{F} \cdot \mathbf{F}^t \text{ with } F_{ij} = \frac{\partial x_i}{\partial X_j} \tag{1.15}$$

where x_i are the coordinates at time t of a point initially at X_i, and $\mathbf{F}^t$ is the transpose of $\mathbf{F}$. The inverse tensor, called the Finger tensor, will be used in Chapter 2:

$$\mathbf{C}^{-1} = \mathbf{F}^{-1} \cdot \left(\mathbf{F}^t\right)^{-1} \tag{1.16}$$

1.1.1.4 Volume Variation During Deformation

Only in extension or compression the strain may result in a variation of the volume. If l_x, l_y, l_z are the dimensions along the three axes, the volume, $\mathcal{V}$, is then

$$\mathcal{V} = l_x l_y l_z \Rightarrow \frac{d\mathcal{V}}{\mathcal{V}} = \frac{dl_x}{l_x} + \frac{dl_y}{l_y} + \frac{dl_z}{l_z} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \tag{1.17}$$

1.1.2 Rate-of-Strain Tensor

For a velocity field $\mathbf{u}(x, y, z)$, the rate-of-strain tensor is defined as the limit:

$$\dot{\boldsymbol{\varepsilon}} = \lim_{dt \to 0} \frac{\boldsymbol{\varepsilon}_t^{t+dt}}{dt} \tag{1.18}$$

where $\boldsymbol{\varepsilon}_t^{t+dt}$ is the deformation tensor between times t and $t + dt$. However, in this time interval the displacement vector is $d\mathbf{U} = \mathbf{u}\, dt$. Hence,

$$\varepsilon_{ij}\,{}_t^{t+dt} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) dt \tag{1.19}$$

where $u_i = (u, v, w)$ are the components of the velocity vector. The components of the rate-of-strain tensor become

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.20}$$

As in the case of $\boldsymbol{\varepsilon}$, this tensor is symmetrical:

$$\dot{\boldsymbol{\varepsilon}} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{bmatrix} \tag{1.21}$$

The diagonal terms are elongational rates; the other terms are shear rates. They are often denoted $\dot{\alpha}$ and $\dot{\gamma}$, respectively.

Remark: Equation (1.20) is the general expression for the components of the rate-of-strain tensor, but its derivation from the expression (1.18) for the strain tensor is correct only if the deformations and the displacements are infinitely small (as in the case of a high-modulus elastic body). For a liquid material, it is not possible, in general, to make use of expression (1.19). Indeed, a liquid experiences very large deformations for which the tensor $\boldsymbol{\varepsilon}$ has no physical meaning. Tensors $\boldsymbol{\Delta}$, $\mathbf{C}$, or $\mathbf{C}^{-1}$ are used instead.

1.1.3 Continuity Equation

1.1.3.1 Mass Balance

Let us consider a volume element of fluid $dx\,dy\,dz$ (Figure 1.4). The fluid density is $\rho(x, y, z, t)$.

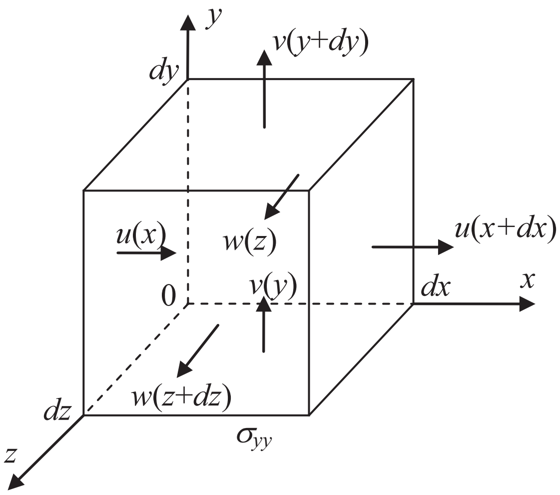

Figure 1.4 Mass balance on a cubic volume element

The variation of mass in the volume element with respect to time is $\frac{\partial \rho}{\partial t} dxdydz$. This variation is due to a balance of mass fluxes across the faces of the volume element:

- In the x direction: $\left(\rho(x+dx)u(x+dx) - \rho(x)u(x)\right)dydz$
- In the y direction: $\left(\rho(y+dy)v(y+dy) - \rho(y)v(y)\right)dzdx$
- In the z direction: $\left(\rho(z+dz)w(z+dz) - \rho(z)w(z)\right)dxdy$

Hence, dividing by $dx\,dy\,dz$ and taking the limits, we get

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \quad (1.22)$$

which can be written through the definition of the divergence as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (1.23)$$

This is the continuity equation.

Remark: This equation can be written using the material derivative $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \mathbf{u}.\nabla\rho$, leading to $\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{u} = 0$.

1.1.3.2 Incompressible Materials

For incompressible materials, ρ is a constant, and the continuity equation reduces to

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0 \tag{1.24}$$

This result can be obtained from the expression for the volume variation in small deformations:

$$\frac{d\mathcal{V}}{\mathcal{V}} = \operatorname{tr} \boldsymbol{\varepsilon} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \tag{1.25}$$

also: $$\frac{1}{\mathcal{V}} \frac{d\mathcal{V}}{dt} = \operatorname{tr} \dot{\boldsymbol{\varepsilon}} = \dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \boldsymbol{\nabla} \cdot \mathbf{u} \tag{1.26}$$

It follows that $$\frac{d\mathcal{V}}{dt} = 0 \Leftrightarrow \operatorname{tr} \dot{\boldsymbol{\varepsilon}} = 0 \Leftrightarrow \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \tag{1.27}$$

1.1.4 Problems

1.1.4.1 Analysis of Simple Shear Flow

Simple shear flow is representative of the rate of deformation experienced in many practical situations. Homogeneous, simple planar shear flow is defined by the following velocity field:

$$u(y) = \dot{\gamma} y \left(\dot{\gamma} = \frac{U}{h} \right) \; ; \; v = 0 \; ; \; w = 0$$

where Ox is the direction of the velocity, Oxy is the shear plane, and planes parallel to Oxz are sheared surfaces; $\dot{\gamma}$ is the shear rate. Write down the expression for the tensor $\dot{\boldsymbol{\varepsilon}}$ for this simple planar shear flow.

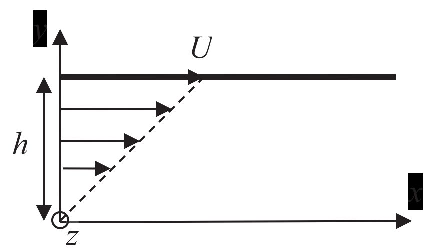

Figure 1.5 Flow between parallel plates

Solution

$$\dot{\boldsymbol{\varepsilon}} = \begin{bmatrix} 0 & \frac{1}{2}\dot{\gamma} & 0 \\ \frac{1}{2}\dot{\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.28}$$

1.1.4.2 Study of Several Simple Shear Flows

One can assume that any flow situation is locally simple shear if, at that given point, the rate-of-strain tensor is given by the above expression (Eq. (1.28)). Then show that all the following flows, encountered in practical situations, are locally simple shear flows. Obtain in each case the directions 1, 2, 3 (equivalent to x, y, z for planar shear) and the expression of the shear rate $\dot{\gamma}$ (use the expressions of $\dot{\boldsymbol{\varepsilon}}$ in cylindrical and spherical coordinates given in Appendix 1, see Section 1.4.1).

1.1.4.2.1 Flow between Parallel Plates (Figure 1.6)

The velocity vector components are $u(y), v = 0, w = 0$.

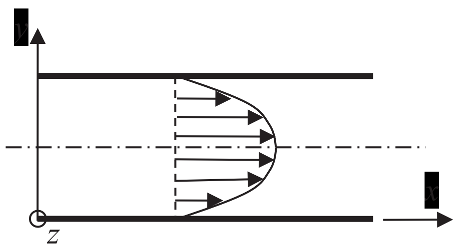

Figure 1.6 Flow between parallel plates

Solution

$$\dot{\boldsymbol{\varepsilon}} = \begin{bmatrix} 0 & \frac{1}{2}\frac{du}{dy} & 0 \\ \frac{1}{2}\frac{du}{dy} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.29}$$

1.1.4.2.2 Flow in a Circular Tube (Figure 1.7)

The components of the velocity vector $\mathbf{u}(r,\theta,z)$ in a cylindrical frame are $u=0, v=0, w=w(r)$.

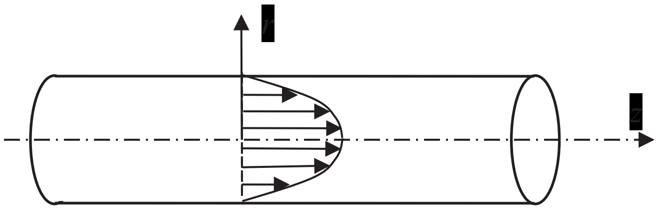

Figure 1.7 Flow in a circular tube

Solution

$$\dot{\boldsymbol{\varepsilon}} = \begin{bmatrix} 0 & 0 & \frac{1}{2}\frac{dw}{dr} \\ 0 & 0 & 0 \\ \frac{1}{2}\frac{dw}{dr} & 0 & 0 \end{bmatrix} \tag{1.30}$$

Directions 1, 2, and 3 are respectively z, r, and θ. The shear rate is $\dot{\gamma} = \frac{dw}{dr}$.

1.1.4.2.3 Flow between Two Parallel Disks

The upper disk is rotating at an angular velocity Ω_0, and the lower one is fixed (Figure 1.8). The velocity field in cylindrical coordinates has the following expression:

$\mathbf{u}(r,\theta,z): u=0, v(r,z), w=0$

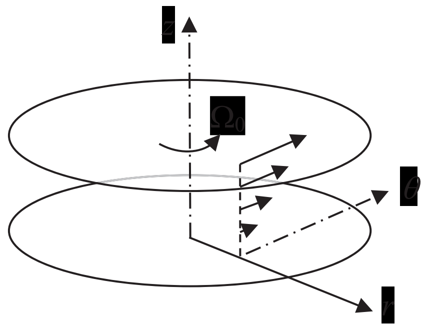

Figure 1.8 Flow between parallel disks

(a) Show that the tensor $\dot{\boldsymbol{\varepsilon}}$ does not have the form defined in Section 1.1.4.1.

(b) The sheared surfaces are now assumed to be parallel to the disks and rotate at an angular velocity $\Omega(z)$. Calculate $v(r, z)$ and show that the tensor $\dot{\boldsymbol{\varepsilon}}$ is a simple shear one.

Solution

(a)

$$\dot{\boldsymbol{\varepsilon}} = \begin{bmatrix} 0 & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial r} - \dfrac{v}{r}\right) & 0 \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial r} - \dfrac{v}{r}\right) & 0 & \dfrac{1}{2}\dfrac{\partial v}{\partial z} \\ 0 & \dfrac{1}{2}\dfrac{\partial v}{\partial z} & 0 \end{bmatrix} \tag{1.31}$$

(b) If $v(r, z) = r\Omega(z)$, then $\dfrac{\partial v}{\partial r} - \dfrac{v}{r} = 0$ and $\dot{\boldsymbol{\varepsilon}}$ is a simple shear tensor. The shear rate is $\dot{\gamma} = \dfrac{dv}{dz} = r\dfrac{d\Omega}{dz}$ and directions 1, 2, and 3 are θ, z, and r, respectively.

1.1.4.2.4 Flow between a Cone and a Plate

A cone of half angle θ_0 rotates with the angular velocity Ω_0. The apex of the cone is on the disk, which is fixed (Figure 1.9). The sheared surfaces are assumed to be cones with the same axis and apex as the cone-and-plate system; they rotate at an angular velocity $\Omega(\theta)$.

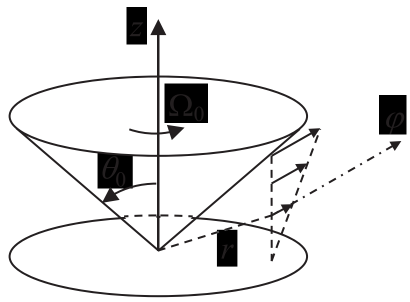

Figure 1.9 Flow in a cone-and-plate system

Solution

In spherical coordinates (r, θ, φ), the velocity vector components are $u = 0$, $v = 0$, and $w = r\sin\theta\,\Omega(\theta)$.

$$\dot{\boldsymbol{\varepsilon}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{2}\sin\theta\dfrac{d\Omega}{d\theta} \\ 0 & \dfrac{1}{2}\sin\theta\dfrac{d\Omega}{d\theta} & 0 \end{bmatrix} \tag{1.32}$$

The shear rate is $\dot{\gamma} = \sin\theta\dfrac{d\Omega}{d\theta}$, and directions 1, 2, and 3 are φ, θ, and r, respectively.

1.1.4.2.5 Couette Flow

A fluid is sheared between the inner cylinder of radius R_1 rotating at the angular velocity Ω_0 and the outer fixed cylinder of radius R_2 (Figure 1.10). The components of the velocity vector $\mathbf{u}(r, \theta, z)$ in cylindrical coordinates are $u = 0$, $v(r)$, and $w = 0$.

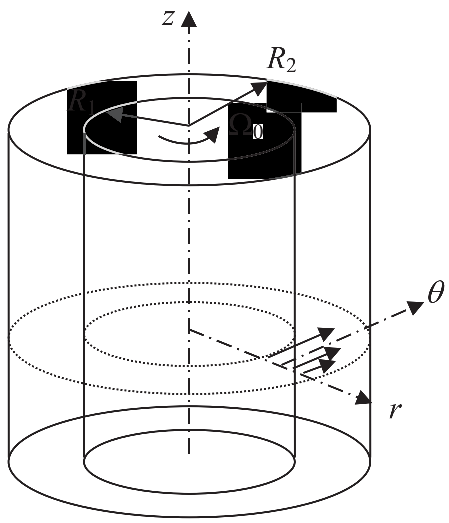

Figure 1.10 Couette flow

Solution

$$\dot{\boldsymbol{\varepsilon}} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{dv}{dr} - \frac{v}{r}\right) & 0 \\ \frac{1}{2}\left(\frac{dv}{dr} - \frac{v}{r}\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.33}$$

The shear rate is $\dot{\gamma} = \frac{dv}{dr} - \frac{v}{r}$, and directions 1, 2, and 3 are θ, r, and z, respectively.

1.1.4.3 Pure Elongational Flow

A flow is purely elongational or extensional at a given point if the rate-of-strain tensor at this point has only nonzero components on the diagonal.

1.1.4.3.1 Simple Elongation

An incompressible parallelepiped specimen of square section is stretched in direction x (Figure 1.11). Then $\dot{\alpha} = \frac{1}{l}\frac{dl}{dt}$ is called the elongation rate in the x-direction. Write down the expression of $\dot{\boldsymbol{\varepsilon}}$.

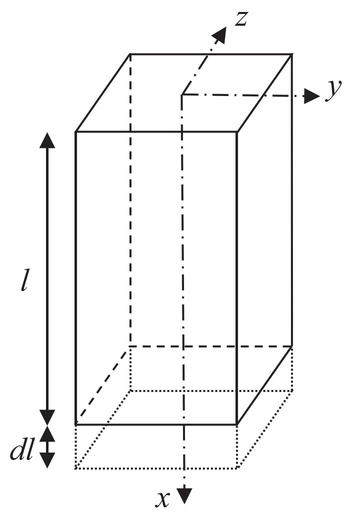

Figure 1.11 Deformation of a specimen in elongation

Solution

Assuming a homogeneous deformation, the velocity vector is $\mathbf{u} = \left(u(x), v(y), w(z)\right)$ and

$$\frac{du}{dx} = \dot{\alpha} = \frac{1}{l}\frac{dl}{dt} \tag{1.34}$$

The sample section remains square during the deformation, so $\frac{dv}{dy} = \frac{dw}{dz}$. Incompressibility implies $\dot{\alpha} + 2\frac{dv}{dy} = 0$. Therefore, $\frac{dv}{dy} = \frac{dw}{dz} = -\frac{\dot{\alpha}}{2}$ and

$$\dot{\boldsymbol{\varepsilon}} = \begin{bmatrix} \dot{\alpha} & 0 & 0 \\ 0 & -\frac{\dot{\alpha}}{2} & 0 \\ 0 & 0 & -\frac{\dot{\alpha}}{2} \end{bmatrix} \tag{1.35}$$

1.1.4.3.2 Biaxial Stretching: Bubble Inflation

The inflation of a bubble of radius R and thickness e small compared to R is considered in Figure 1.12.

a) Write the rate-of-strain components in the r, θ, φ directions.
b) Write the continuity equation for an incompressible material and integrate it.
c) Show the equivalence between the continuity equation and the volume conservation.

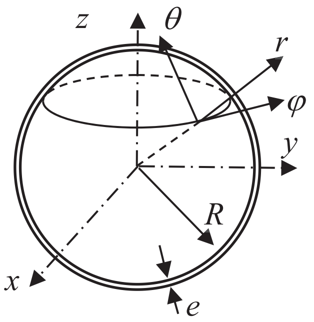

Figure 1.12 Bubble inflation

Solution

(a) The bubble is assumed to remain spherical and to deform homogeneously so that the shear components are zero. The rate-of-strain components are as follows:

- In the thickness (r) direction: $\dot{\varepsilon}_{rr} = \frac{1}{e}\frac{de}{dt}$
- In the θ-direction: $\dot{\varepsilon}_{\theta\theta} = \frac{1}{2\pi R}\frac{d(2\pi R)}{dt} = \frac{1}{R}\frac{dR}{dt}$
- In the φ-direction: $\dot{\varepsilon}_{\varphi\varphi} = \frac{1}{2\pi R\sin\theta}\frac{d(2\pi R\sin\theta)}{dt} = \frac{1}{R}\frac{dR}{dt}$

(b) For an incompressible material, $\frac{1}{e}\frac{de}{dt} + \frac{2}{R}\frac{dR}{dt} = 0$, which can be integrated to obtain $R^2 e = \text{cst}$.

(c) This is equivalent to the global volume conservation: $4\pi R^2 e = 4\pi R_0^2 e_0$.

1.2 Stresses and Force Balances

1.2.1 Stress Tensor

1.2.1.1 Phenomenological Definitions

1.2.1.1.1 Extension (or Compression) (Figure 1.13)

An extension force applied on a cylinder of section S induces a normal stress $\sigma_n = F/S$.

Figure 1.13 Stress in extension

1.2.1.1.2 Simple Shear (Figure 1.14)

A force tangentially applied to a surface S yields a shear stress $\tau = F/S$.

The units of the stresses are those of pressure: pascals (Pa).

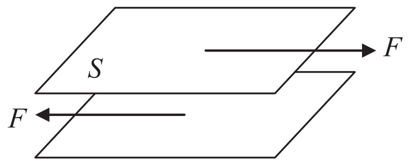

Figure 1.14 Stress in simple shear

1.2.1.2 Stress Vector

Let us consider, in a more general situation, a surface element dS in a continuum. The part of the continuum located on one side of dS exerts on the other part a force $d\mathbf{F}$. As the interactions between both parts of the continuum are at small distances, the stress vector $\mathbf{T}$ at a point O on this surface is defined as the limit:

$$\mathbf{T} = \lim_{dS \to 0} \frac{d\mathbf{F}}{dS} \tag{1.36}$$

At point O, the normal to the surface is defined by the unit vector, $\mathbf{n}$, in the outward direction, as illustrated in Figure 1.15.

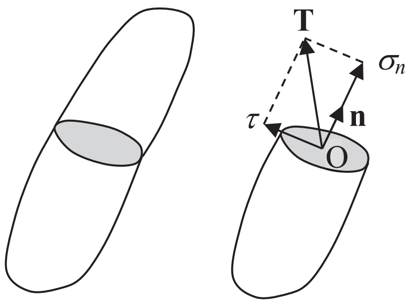

Figure 1.15 Stress applied to a surface element

The stress components can be obtained from projections of the stress vector:

- Projection on $\mathbf{n}$: $\sigma_n = \mathbf{T} \cdot \mathbf{n}$
  where σ_n is the normal stress (in extension, $\sigma_n > 0$; in compression, $\sigma_n < 0$).
- Projection on the surface: τ is the shear stress.

1.2.1.3 Stress Tensor

The stress vector cannot characterize the state of stresses at a given point since it is a function of the orientation of the surface element, that is, of **n**. Thus, a tensile force induces a stress on a surface element perpendicular to the orientation of the force, but it induces no stress on a parallel surface element (Figure 1.16).

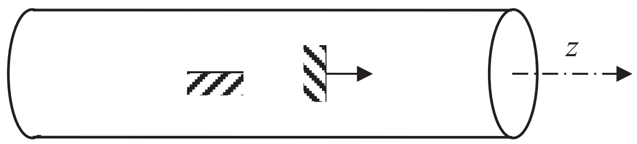

Figure 1.16 Stress vector and surface orientation

The state of stresses is in fact characterized by the relation between **T** and **n** and, as we will see, this relation is tensorial. Let us consider an elementary tetrahedron *OABC* along the axes *Oxyz* (Figure 1.17): the *x*, *y*, and *z* components of the unit normal vector to the *ABC* plane are the ratios of the surfaces *OAB*, *OBC*, and *OCA* to *ABC*:

$$n_x = \frac{OBC}{ABC} \quad n_y = \frac{OCA}{ABC} \quad n_z = \frac{OAB}{ABC}$$

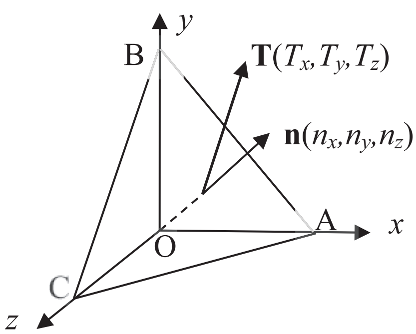

Figure 1.17 Stresses exerted on an elementary tetrahedron

Let us define the components of the stress tensor in the following table:

Projection on	of the stress vector exerted on the face normal to		
	Ox	*Oy*	*Oz*
Ox	σ_{xx}	σ_{xy}	σ_{xz}
Oy	σ_{yx}	σ_{yy}	σ_{yz}
Oz	σ_{zx}	σ_{zy}	σ_{zz}

The net surface forces acting along the three directions of the axes are as follows:

$$T_x(ABC) - \sigma_{xx}(OBC) - \sigma_{xy}(OAC) - \sigma_{xz}(OAB)$$
$$T_y(ABC) - \sigma_{yx}(OBC) - \sigma_{yy}(OAC) - \sigma_{yz}(OAB)$$
$$T_z(ABC) - \sigma_{zx}(OBC) - \sigma_{zy}(OAC) - \sigma_{zz}(OAB)$$

with OA, OB, OC being of the order of d; the surfaces OAB, OBC, and OCA are of the order of d^2; and the volume $OABC$ is of the order of d^3. The surface forces are of the order of Td^2 and the volume forces of the order of Fd^3 (e.g., $F = \rho g$ for the gravitational force per unit volume).

When the dimension d of the tetrahedron tends to zero, the volume forces become negligible compared with the surface forces, and the net forces, as expressed above, are equal to zero. Hence, in terms of the components of **n**:

$$\begin{aligned} T_x &= \sigma_{xx}n_x + \sigma_{xy}n_y + \sigma_{xz}n_z \\ T_y &= \sigma_{yx}n_x + \sigma_{yy}n_y + \sigma_{yz}n_z \\ T_z &= \sigma_{zx}n_x + \sigma_{zy}n_y + \sigma_{zz}n_z \end{aligned} \tag{1.37}$$

This result can be written in tensorial notation as

$$\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n} \tag{1.38}$$

where $\boldsymbol{\sigma}$ is the stress tensor, which contains three normal components and six shear components defined for the three axes. As in the case of the strain, the state of the stresses is described by a tensor.

1.2.1.4 Isotropic Stress or Hydrostatic Pressure

The hydrostatic pressure translates into a stress vector that is in the direction of **n** for any orientation of the surface:

$$\mathbf{T} = -p\mathbf{n} \tag{1.39}$$

The corresponding tensor is proportional to the unit tensor **I**:

$$\boldsymbol{\sigma} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix} = -p\mathbf{I} \tag{1.40}$$

1.2.1.5 Deviatoric Stress Tensor

For any general state of stresses, the pressure can be defined in terms of the trace of the stress tensor as

$$p = -\frac{1}{3}\mathrm{tr}\,\boldsymbol{\sigma} = -\frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} \tag{1.41}$$

The pressure is independent of the axes since the trace of the stress tensor is an invariant (see Appendix 2, see Section 1.4.2). It could be positive (compressive state) or relatively negative (extensive state, possibly leading to cavitation problems in a liquid).

The stress tensor can be written as a sum of two terms, the pressure term and a traceless stress term, called the deviatoric stress tensor $\boldsymbol{\sigma}'$:

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\sigma}' \tag{1.42}$$

Examples

- *Uniaxial extension* (or compression):

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow p = -\frac{\sigma_{11}}{3}, \quad \boldsymbol{\sigma}' = \begin{bmatrix} \frac{2\sigma_{11}}{3} & 0 & 0 \\ 0 & -\frac{\sigma_{11}}{3} & 0 \\ 0 & 0 & -\frac{\sigma_{11}}{3} \end{bmatrix} \tag{1.43}$$

- *Simple shear* under a hydrostatic pressure p:

$$\boldsymbol{\sigma} = \begin{bmatrix} -p & \tau & 0 \\ \tau & -p & 0 \\ 0 & 0 & -p \end{bmatrix} \Rightarrow \boldsymbol{\sigma}' = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.44}$$

More generally, we will see that the stress tensor can be decomposed into an isotropic arbitrary part denoted as $p'\mathbf{I}$, and a tensor called the extra-stress tensor $\boldsymbol{\sigma}'$. The expressions of the constitutive equations in Chapter 2 will use either the deviatoric part of the stress tensor $\boldsymbol{\sigma}'$ for viscous behaviors or the extra-stress tensor $\boldsymbol{\sigma}'$ for viscoelastic behaviors (in this case, $\boldsymbol{\sigma}'$ is no longer a deviator, and p' is not the hydrostatic pressure).

1.2.2 Equation of Motion

1.2.2.1 Force Balances

Considering an elementary volume of material with a characteristic dimension d:

- The surface forces are of the order of d^2, but the definition of the stress tensor is such that their contribution to a force balance is nil.
- The volume forces (gravity, inertia) are of the order of d^3, and they must balance the derivatives of the surface forces, which are also of the order of d^3.

We will write that the resultant force is nil (Figure 1.18).

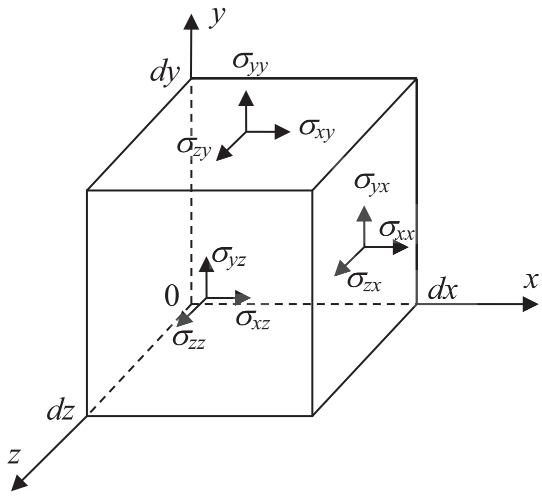

Figure 1.18 Balance of forces exerted on a volume element

The forces acting on a volume element *dx dy dz* are the following:

- The mass force (generally gravity): **F** *dx dy dz*
- The inertial force: $\rho\boldsymbol{\gamma}\, dx\, dy\, dz = \rho\, (d\mathbf{u}/dt)\, dx\, dy\, dz$
- The net surface force exerted by the surroundings in the *x*-direction:

  $$\left[\sigma_{xx}(x+dx)-\sigma_{xx}(x)\right]dydz+\left[\sigma_{xy}(y+dy)-\sigma_{xy}(y)\right]dzdx+\left[\sigma_{xz}(z+dz)-\sigma_{xz}(z)\right]dxdy$$

  and similar terms for the *y* and *z*-directions.

Dividing by *dx dy dz* and taking the limits, we obtain for the *x*, *y*, and *z* components:

$$\begin{aligned} F_x-\rho\gamma_x+\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}&=0\\ F_y-\rho\gamma_y+\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{yz}}{\partial z}&=0\\ F_z-\rho\gamma_z+\frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}&=0 \end{aligned} \tag{1.45}$$

The derivatives of σ_{ij} are the components of a vector, which is the divergence of the tensor $\boldsymbol{\sigma}$. Equation (1.45) may be written as

$$\nabla\cdot\boldsymbol{\sigma}+\mathbf{F}-\rho\boldsymbol{\gamma}=0 \tag{1.46}$$

This is the equation of motion, also called the dynamic equilibrium. It is often convenient to express the stress tensor as the sum of the pressure and the deviatoric stress:

$$-\nabla p+\nabla\cdot\boldsymbol{\sigma}'+\mathbf{F}-\rho\boldsymbol{\gamma}=0 \tag{1.47}$$

1.2.2.2 Torque Balances

Let us consider a small volume element of linear dimension d; the mass forces of the order of d^3 induce torques of the order of d^4. There is no mass torque, which would result in torques of the order of d^3 (as in the case of a magnetic medium). Finally, the surface forces of the order of d^2 induce torques of the order of d^3, so only the net torque resulting from these forces must be equal to zero.

If we consider the moments about the z-axis (Figure 1.19), only the shear stresses σ_{xy} and σ_{yx} on the upper (U) and lateral (L) surfaces of the element $dx\ dy\ dz$ lead to torques. They are obtained by taking the following vector products:

$$\sigma_{xy}: \begin{pmatrix} 0 \\ dy \\ 0 \end{pmatrix} \times \begin{pmatrix} \sigma_{xy} dxdz \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\sigma_{xy} dxdydz \end{pmatrix} \tag{1.48}$$

$$\sigma_{yx}: \begin{pmatrix} dx \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ \sigma_{yx} dydz \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sigma_{yx} dxdydz \end{pmatrix} \tag{1.49}$$

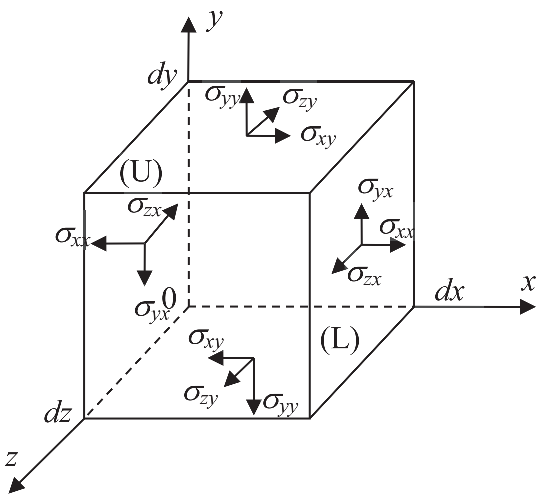

Figure 1.19 Torque balance on a volume element

A torque balance, in the absence of a mass torque, yields $\sigma_{xy} = \sigma_{yx}$. In a similar way, $\sigma_{yz} = \sigma_{zy}$ and $\sigma_{zx} = \sigma_{xz}$. The absence of a volume torque then implies the symmetry of the stress tensor. Therefore, as for the strain tensor $\boldsymbol{\varepsilon}$, the stress tensor has only six independent components (three normal and three shear components).

Subject Index